Appendix. Countable and uncountable sets

Definition A1

A set is *countable* if it can be put in one–one correspondence with the set of natural numbers. In other words, a set A is countable if there is a bijection $f: D_N \to A$.

Notice that the elements of a countable set may be written out in a list, and that the bijection given in the definition provides a method for doing this, namely enumerating $f(0), f(1), f(2), \ldots$

Trivially, the natural numbers constitute a countable set.

Proposition A2

If A and B are countable sets then there is a bijection between A and B. Conversely, if A is a countable set and there is a bijection between A and B, then B is countable.

Proof. Let $f: D_N \to A$ and $g: D_N \to B$ be bijections. Then $g \circ f^{-1}$ is a bijection from A to B.

For the converse, let $f: D_N \to A$ be a bijection, so that A is countable, and suppose that there is a bijection h from A to B. Then $h \circ f$ is a bijection from D_N to B, and so B is countable.

Proposition A3

Any infinite subset of a countable set is countable.

Proof. Let A be a countable set, let $f: D_N \to A$ be a bijection and let B be an infinite subset of A. Then $f(0), f(1), f(2), \ldots$ is a list of all the members of A. Delete from this list all elements which are not members of B. What remains is a list (infinite) of the members of B. A bijection $g: D_N \to B$ can now be defined by

$$g(n) = \text{the } (n+1)\text{th member of the new list } (n \in D_N).$$

(It must be the $(n+1)$th member because $g(0)$ is the first, $g(1)$ is the second, and so on.)

Proposition A4

An infinite set A is countable if and only if there is an injection $h: A \to D_N$.

Proof. If A is countable then there is a bijection $D_N \to A$, whose inverse is certainly an injection $A \to D_N$.

Conversely, suppose that there is an injection $h: A \to D_N$. $h(A) \subseteq D_N$ and $h(A)$ is infinite, since h is one–one. By Proposition A3, $h(A)$ is countable. Let $g: D_N \to h(A)$ be a bijection. The composition $h^{-1} \circ g$ is then a bijection from D_N to A, and A is countable.

▷ This last result is usually the most convenient to use in a demonstration that a particular set is countable, and we shall see applications of it shortly.

Proposition A5

The union of two disjoint countable sets is countable.

Proof. Let A and B be disjoint countable sets and let $f: D_N \to A$ and $g: D_N \to B$ be bijections. Define $h: D_N \to A \cup B$ as follows:

$$h(n) = \begin{cases} f(\tfrac{1}{2}n) & \text{if } n \text{ is even,} \\ g(\tfrac{1}{2}n - \tfrac{1}{2}) & \text{if } n \text{ is odd.} \end{cases}$$

It is easy to verify that h is a bijection, and so $A \cup B$ is countable. (h gives the list $f(0)$, $g(0)$, $f(1)$, $g(1)$, $f(2)$, ... of the elements of $A \cup B$.)

Corollary A6

The union of any finite collection of disjoint countable sets is countable.

Proof. The proof is by induction on the number of sets in the collection.

Base step: The union of two disjoint countable sets is countable, by the proposition.

Induction step: Let $n > 2$ and let $A_1, \ldots A_n$ be disjoint countable sets. Suppose as induction hypothesis that a union of $n-1$ disjoint countable sets is countable. Then $A_1 \cup A_2 \cup \ldots \cup A_{n-1}$ is countable (and disjoint from A_n). By the proposition, then, the set $(A_1 \cup \ldots \cup A_{n-1}) \cup A_n$ is countable.

The result follows by the principle of mathematical induction.

Remark. The requirement that the sets be disjoint can be dispensed with. A demonstration of this is left as an exercise for the reader.

Question. Do there exist sets which are neither finite nor countable? We know from the above that D_N and all of its subsets are finite or countable. The answer is given in the following proposition.

Proposition A7

The set of all subsets of D_N is infinite and not countable.

Proof. Denote by $P(D_N)$ the set of all subsets of D_N. $P(D_N)$ is clearly infinite. Suppose that it is countable, and let $f: D_N \to P(D_N)$ be a bijection. Then for each $n \in D_N$, $f(n)$ is a subset of D_N. Let

$$B = \{k \in D_N : k \notin f(k)\}.$$

B is certainly a subset of D_N (possibly empty and possibly all of D_N). Also $B \neq f(n)$ for any $n \in D_N$. For suppose that $B = f(n)$. If $n \in f(n)$ then $n \in B$, since $B = f(n)$, but $n \notin B$ by the definition of B. If $n \notin f(n)$, then $n \notin B$, since $B = f(n)$, but $n \in B$ by definition of B. Either way we reach a contradiction. Therefore $B \neq f(n)$ for any $n \in D_N$, and so f is not a bijection between D_N and $P(D_N)$. This contradicts our original assumption, and hence $P(D_N)$ is not countable.

Corollary A8

The set of all functions on D_N is not countable.

Proof. For each subset A of D_N, define a function $C_A: D_N \to D_N$ (the characteristic function of A) as follows.

$$C_A(n) = \begin{cases} 0 & \text{if } n \in A, \\ 1 & \text{if } n \notin A. \end{cases}$$

The correspondence between sets A and functions C_A is a bijection from $P(D_N)$ to a *subset* of the set of functions on D_N. $P(D_N)$ is not countable, so the set of functions on D_N has a subset which is not countable. (If there is a bijection between two sets and one of them is countable, then the other is countable.) If the set of functions on D_N were countable, then any infinite subset would be countable (by Proposition A3). Hence, since it has an uncountable subset, the set of all functions on D_N is not countable.

Corollary A9

The set of all relations on D_N is not countable.

*Proof.*The set of relations includes the set of functions. By an argument similar to the above, then, the set of relations cannot be countable.

▷ In the text we use the result that if an uncountable set has a countable subset, then that subset must be a proper subset. This should now be clear, since a set cannot be both countable and uncountable.

We also use the result that the set of *wf*s. in a given symbolic language was countable. There are some general results which lead us to see why this is so.

Proposition A10

Let A be a countable set. The collection of all *finite* subsets of A is a countable set.

Proof. Let $f: D_N \to A$ be a bijection. We may define an injection g from the set of all finite subsets of A into D_N as follows. Let F be a finite subset of A. Then $f^{-1}(F)$ is a finite subset of D_N. Let $g(F)=$ the product of the primes p_n, for $n \in f^{-1}(F)$. (Here p_i denotes the ith odd prime, for $i>0$, and $p_0=2$.) g is one–one since the same product of primes cannot possibly arise from two different sets F, and two different products of primes cannot be equal. By Proposition A4, then, the set of all finite subsets of D_N is countable.

Proposition A11

Let A be a countable set. Then the set of all finite *sequences of* elements of A is a countable set.

Proof. We can make use of the properties of prime numbers here in a slightly different way. Let $f: D_N \to A$ be a bijection. Define an injection h from the set of all finite sequences of elements of A into D_N as follows. If $u_0, u_1, \ldots, u_k \in A$, let

$$h(u_0, u_1, \ldots, u_k) = p_0^{f^{-1}(u_0)} \times p_1^{f^{-1}(u_1)} \times \ldots \times p_k^{f^{-1}(u_k)},$$

where the p_i are as in the previous proof. h is one–one because f is a bijection and because of the uniqueness of prime power decomposition. By Proposition A4, then, the set of all finite sequences of elements of A is countable.

▷ This proposition has as a corollary the result which we need about formal languages. All of our formal languages have alphabets of symbols which are countable sets. (Demonstration of this requires use of Corollary A6.) The set of *wf*s. of a formal language $\mathscr{L}$ is a subset of the set of all finite sequences of symbols from the alphabet of $\mathscr{L}$. This set of finite sequences is countable, so any subset of it is also countable (or finite). But we know that the set of all *wf*s. is always infinite, and so we have our result.